

# AN EXACT MEMBRANE QUANTIZATION FROM $W_\infty$ SYMMETRY

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March,1995

## ABSTRACT

An exact quantization of the spherical membrane moving in flat target spacetime backgrounds is performed. Crucial ingredients are the exact integrability of the  $3D$   $SU(\infty)$  continuous Toda equation and the quasi-finite highest weight irreducible representations of  $W_\infty$  algebras. Both continuous and discrete energy levels are found. The latter are found for periodic-like solutions. Membrane wavefunctionals solutions are found in terms of Bessel's functions and plausible relations to singleton field theory are outlined.

PACS : 0465.+e; 02.40.+m

## I. Introduction

Recently, [1] exact instanton solutions to  $D = 11$  spherical membranes moving in flat target spacetime backgrounds were constructed. The starting point was dimensionally-reduced Super Yang-Mills theories based on the infinite dimensional  $SU(\infty)$  algebra. The latter algebra is isomorphic to the area-preserving diffeomorphisms of the sphere. In this fashion the super Toda molecule equation was recovered preserving one supersymmetry out of the  $N = 16$  expected. The expected critical target spacetime dimensions for the (super) membrane,  $D = 27(11)$ , was closely related to that of non-critical (super)  $W_\infty$  string theories. A BRST analysis revealed that the spectrum of the membrane must have a relationship to the first unitary minimal model of a  $W_N$  algebra adjoined to a critical  $W_N$  string in the  $N \rightarrow \infty$  limit [1]. It is the purpose of this work to push this connection forward.

In **II** we briefly review the contents of [1] and show the crucial role that the continuous Toda equation, a  $D = 3$  integrable field theory, [2,3] has in the membrane quantization program. In the final section we quantize the continuous Toda equation and establish the relationship between this quantization program and the construction of highest weight representations of  $W_\infty$  algebras [10,11]. This enables us to establish the connection between the membrane quantization and representations of  $W_\infty$  algebras via the continuous Toda theory. Brief comments about black holes, singleton-field theory and universal string theory are presented at the conclusion.

## II

Based on the observation that the spherical membrane moving in  $D$  spacetime dimensions, in the light-cone gauge, is essentially equivalent to a  $D - 1$  Yang-Mills theory, dimensionally reduced to one time dimension, of the  $SU(\infty)$  group (see Duff [20] for a review) we look for solutions of the  $D = 10$  Yang-Mills equations (dimensionally-reduced to one temporal dimension).

In [1] we obtained solutions to :

$$\partial_a F_{ab} + [A_a, F_{ab}] = 0. A_a^\alpha T_\alpha \rightarrow A_a(x^b; q, p). [A_a, A_b] \rightarrow \{A_a, A_b\}_{q,p}. \quad (1)$$

with  $a, b, \dots = 8$  being the transverse indices to the membrane ; we performed the  $10 = 2 + 8$  split of the original  $D = 10$  YM equations. After the dimensional reduction to one dimension we found that the following  $D = 10$  YM potentials,  $\mathcal{A}$  are one class of solutions to the original equations :

$$\mathcal{A}_1 = p_1 A_1. \mathcal{A}_5 = p_2 A_1. \mathcal{A}_3 = p_1 A_3. \mathcal{A}_7 = p_2 A_3. \quad (2a)$$

$$\mathcal{A}_2 = p_1 A_2. \mathcal{A}_6 = p_2 A_2. \mathcal{A}_0 = \mathcal{A}_4 = \mathcal{A}_8 = \mathcal{A}_9 = 0. \quad (2b)$$

where  $p_1, p_2$  are constants and  $A_1, A_2, A_3$  are functions of  $x_0, q, p$  only and obey the  $SU(\infty)$  Nahm's equations :

$$\epsilon_{ijk} \frac{\partial A_k}{\partial x_0} + \{A_i, A_j\}_{q,p} = 0. i, j, k = 1, 2, 3. \quad (3)$$

Nahm's equations were obtained from reductions of Self Dual Yang-Mills equations in  $D = 4$ . The temporal variable  $x_0 = p_1 X_0 + p_2 X_4$ . We refer to [1] for details.

Expanding  $A_y = \sum A_{yl} Y_{l,+1}$ .  $A_{\bar{y}} = \sum A_{\bar{y}l} Y_{l,-1}$ ; and  $A_3$  in terms of  $Y_{l0}$ , the ansatz which allows to recast the  $SU(\infty)$  Nahm's equations as a Toda molecule equation is [1] :

$$\{A_y, A_{\bar{y}}\} = -i \sum_{l=1}^{\infty} \exp[K_{ll'} \theta_{l'}] Y_{l0}(\sigma_1, \sigma_2). A_3 = - \sum_{l=1}^{\infty} \frac{\partial \theta_l}{\partial \tau} Y_{l0}. \quad (4)$$

with  $A_y = \frac{A_1 + i A_2}{\sqrt{2}}$ .  $A_{\bar{y}} = \frac{A_1 - i A_2}{\sqrt{2}}$ .

Hence, Nahm's equations become :

$$-\frac{\partial^2 \theta_l}{\partial \tau^2} = e^{K_{ll'} \theta_{l'}}. l, l' = 1, 2, 3, \dots \quad (5)$$

This is the  $SU(N)$  Toda molecule equation in Minkowski form. The  $\theta_l$  are the Toda fields where  $SU(2)$  has been embedded minimally into  $SU(N)$ .  $K_{ll'}$  is the Cartan matrix which in the continuum limit becomes  $\delta''(t - t')$  [2,3]. The continuum limit of (5) is

$$-\frac{\partial^2 \theta(\tau, t)}{\partial \tau^2} = \exp \left[ \int dt' \delta''(t - t') \theta(\tau, t') \right]. \quad (6)$$

Or in alternative form :

$$-\frac{\partial^2 \Psi(\tau, t)}{\partial \tau^2} = \int \delta''(t - t') \exp[\Psi(\tau, t')] dt' = \frac{\partial^2 e^\Psi}{\partial t^2}. \quad (7)$$

if one sets  $K_{ll'} \theta_{l'} = \Psi_l$ . The last two equations are the dimensional reduction of the  $3D \rightarrow 2D$  continuous Toda equation given by Leznov and Saveliev :

$$\frac{\partial^2 u}{\partial \tau^2} = -\frac{\partial^2 e^u}{\partial t^2}. i\tau = r = z + \bar{z}. \quad (8)$$

This equation was obtained by Boyer, Finley and Plebanski, as rotational Killing symmetry reductions of Self-Dual Gravity in  $D = 4$ .

In [1] we established the correspondence between the target space-times of non-critical  $W_\infty$  strings and that of membranes in  $D = 27$  dimensions. The supersymmetric case was also discussed and  $D = 11$  was retrieved. It was shown in [12] that the effective induced action of  $W_N$  gravity in the conformal gauge takes the form of a Toda action for the scalar fields and the  $W_N$  currents take the familiar free field form. The same action can be obtained from a constrained  $WZNW$  model. Each of these Toda actions possesses a  $W_N$  symmetry. The authors [12,13] coupled  $W_N$  matter to  $W_N$  gravity in the conformal gauge, and integrating out the matter fields, they arrived at the induced effective action which was precisely the same as the Toda action. Non-critical  $W_N$  strings are constructed the same way. The matter and Liouville sector of the  $W_N$  algebra can be realized in terms of  $N - 1$  scalars,  $\phi_k, \sigma_k$  respectively. These realizations in general have background charges which are fixed by the Miura transformations [5,6]. The non-critical string is characterized by the central charges of the matter and Liouville sectors,  $c_m, c_L$ . To achieve a nilpotent BRST operator these central charges must satisfy :

$$c_m + c_L = -c_{gh} = 2 \sum_{s=2}^N (6s^2 - 6s + 1) = 2(N-1)(2N^2 + 2N + 1). \quad (9)$$

In the  $N \rightarrow \infty$  limit a zeta function regularization yields  $c_m + c_L = -2$ .

We were able to show that the critical membrane background in  $D = 27$  was the same as that of a non-critical  $W_\infty$  string background if one adjoined the first unitary minimal model of the  $W_N$  algebra to that of a critical  $W_N$  string spectrum in the  $N \rightarrow \infty$  limit. In particular :

$$c_{eff} = D = 1 - 12x^2 = (1 - 12x_o^2) + c_{m_o} = 26 - (1 - \frac{6}{(N+1)(N+2)}) + 2\frac{N-1}{N+2} \Rightarrow D = 27. \quad (10)$$

The value for the total central charge of the matter sector is  $c_m = 2 + \frac{1}{24}$  after a zeta function regularization. That of the Liouville sector is  $c_L = -4 - \frac{1}{24}$ . These values bear an important connection to the notion of Unifying  $W$  algebras [4]. It happens that when the central charges have for values at  $c(n) = \frac{2(n-1)}{n+2}; \frac{2(1-2n)}{n-2}; -1 - 3n$ , there exists a Unifying Quantum Casimir  $W$  algebra :

$$\mathcal{WA}_{n-1} \leftrightarrow \mathcal{W}(2, 3, 4, 5) \sim \frac{\hat{sl}(2, R)_n}{\hat{U}(1)}. \quad (11)$$

in the sense that these algebras truncate at degenerate values of the central charge to a smaller algebra.

We see that the value  $c_m$  after regularisation corresponds to the central charge of the first unitary minimal model of  $WA_{n-1}$  after  $n$  is analytically continued to a negative value  $n = -146 \Rightarrow 2(n-1)/(n+2) = 2 + 1/24$ . The value of  $c_L$  does not correspond to a minimal model but it also corresponds to a special value of  $c$  where the  $WA_{n-1}$

algebra truncates to that of the unifying coset :  $c(n) = 2(1 - 2n)/(n - 2) = -4 - 1/24$  for  $n = 146$ . In virtue of the quantum equivalence between negative rank  $A_{n-1}$  Lie algebras with  $n = 146 \rightarrow -n = -146$ ;  $c = 2 + \frac{1}{24}$  and  $\mathcal{W}(2, 3, 4, 5) \sim W_\infty$  at  $c = 2$ ; i.e. the Hilbert spaces are isomorphic, we can study the spectrum of non-critical  $\mathcal{W}(2, 3, 4, 5)$  strings and claim that it ought to give very relevant information concerning the membrane's spectrum. Unfortunately, non-critical  $\mathcal{W}(2, 3, 4, 5)$  strings are prohibitively complicated. One just needs to look into the cohomology of ordinary critical  $W_{2,s}$  strings to realize this [5]. Nevertheless there is a way in which one can circumvent this problem. The answer lies in the integrability property of the continuous Toda equation [2,3] and the recently constructed quasi-finite highest weight irreducible representations of  $W_{1+\infty}, W_\infty$  algebras [10,11]. For this purpose one needs to compute explicitly the value of the coupling constant appearing in the exponential potential of the Quantum Toda theory [1]. The latter is conformally invariant and the conformally improved stress energy tensor obeys a Virasoro algebra with an adjustable central charge whose value depends on the coupling constant  $\beta$  appearing in the potential [15]. This value coincides precisely with the one obtained from a Quantum Drinfeld-Sokolov reduction of the  $SL(N, R)$  Kac-Moody algebra at level  $k$  :

$$\beta = \frac{1}{\sqrt{k+N}}. \quad c(\beta) = (N-1) - 12|\beta\rho - (1/\beta)\tilde{\rho}|^2. \quad (12)$$

where  $\rho, \tilde{\rho}$  are the Weyl vectors of the (dual)  $A_N$  algebra. We see [1,12] that one can now relate the value of the background charge  $x$  in (10) and  $\beta$  when  $k = -\infty$ .  $N = \infty$ .  $k + N = \text{constant}$  :

$$2x^2 = (-13/3) = \left(\frac{1}{\sqrt{k+N}} - \sqrt{k+N}\right)^2 = \left(\beta - \frac{1}{\beta}\right)^2 \Rightarrow \beta^2 = \frac{-7 + (-)\sqrt{13}}{6}.. \quad (13)$$

so  $\beta$  is purely imaginary. This should not concern us. There exist integrable field theories known as Affine Toda theories whose coupling is imaginary but possesses soliton solutions with real energy and momentum [9].

### III

Having reviewed the essential results of [1] permits us to look for classical solutions to the continuous Toda equation and to implement the Quantization program presented in [2]. The general solution to (8) depending on two variables, say  $r \equiv z_+ + z_-$  and  $t$  (not to be confused with time) was given by [2]. The solution is determined by two arbitrary functions,  $\varphi(t)$  and  $d(t)$ . It is :

$$\exp[-x(r, t)] = \exp[-x_o(r, t)] \left\{ 1 + \sum_{n>1} (-1)^n \sum_{\omega} \int \int \dots \exp[r \sum_{m=1}^n \varphi(t_m)] \Pi dt_m d(t_m) \right. \\ \left. \left[ \sum_{p=m}^n \varphi(t_p) \right]^{-1} \left[ \sum_{q=m}^n \varphi(t_{\omega(q)}) \right]^{-1} \cdot [\epsilon_m(\omega) \delta(t-t_m) - \sum_{l=1}^{m-1} \delta''(t_l-t_m) \theta[\omega^{-1}(m) - \omega^{-1}(l)]] \right\}. \quad (14)$$

with :  $\rho_o = \partial^2 x_o / \partial t^2 = r\varphi(t) + \ln d(t)$ . This defines the boundary values of the solution  $x(r, t)$  in the asymptotic region  $r \rightarrow \infty$ .

$\theta$  is the Heaviside step-function.  $\omega$  is any permutation of the indices from  $[2, \dots, n] \rightarrow [j_2, \dots, j_n]$ .

$\omega(1) \equiv 1$ .  $\epsilon_m(\omega)$  is a numerical coefficient. See [2] for details.

An expansion of (14) yields :

$$\exp[-x] = \exp[-x_o] \{1 - \mu + \frac{1}{2}\mu^2 + \dots\}. \quad (15)$$

where :

$$\mu \equiv \frac{d(t)\exp[r\varphi(t)]}{\varphi^2}. \quad (16)$$

The solution to the Quantum  $A_\infty$  (continuous) Toda chain can be obtained by taking the continuum limit of the general solution to the finite nonperiodic Toda chain associated with the Lie algebra  $A_N$  in the  $N \rightarrow \infty$  limit. This is performed by taking the continuum limit of eqs-(82-86) of [3] :

$$\varphi_i \rightarrow x_o(r, t). \quad \psi_{j_s} \rightarrow \partial^2 x_o / \partial t_s^2 = r\varphi(t_s) + \ln d(t_s). \quad (17)$$

In the  $r \rightarrow \infty$  limit the latter tends to  $r\varphi(t_s)$ .

$$\sum_{j_1 j_2 \dots j_n} \rightarrow \int \dots dt_1 dt_2 \dots dt_n. \mathcal{P}^1 \rightarrow [\sum \varphi(t_p) + O(h)]^{-1} \cdot \mathcal{P}^2 \rightarrow [\sum \varphi(t_{\omega(q)}) + O(h)]^{-1}. \quad (18)$$

Therefore, one just has to write down the quantum corrections to the two factors  $[\sum \varphi]^{-1}$  of eq-(14). One must replace the first factor by a summation from  $p = m$  to  $p = n$  of terms like :

$$[\varphi(t_p) + O(h)] \rightarrow \varphi(t_p) - (ih/2) \left[ \frac{1}{w(t_p)} \right]_{,t_p t_p} - ih \sum_{l=p+1}^n \left[ \frac{1}{w(t_l)} \right]_{,t_l t_l}. \quad (19)$$

and the second factor by a summation from  $q = m$  to  $q = n$  of terms like :

$$\begin{aligned} [\varphi(t_{\omega(q)}) + O(h)] &\rightarrow [eq (19) : p \rightarrow \omega(q)] + ih - ih \sum_{l=1}^{q-1} \left[ \frac{1}{w(t_{\omega(l)})} \right]_{,t_{\omega(l)} t_{\omega(l)}} \\ &\quad + ih \sum_{l=q+1}^n \left[ \frac{1}{w(t_{\omega(l)})} \right]_{,t_{\omega(l)} t_{\omega(l)}} \end{aligned} \quad (20)$$

where  $w(t)$  is a positive function that is the continuum limit of eqs-(30,34) of [3]. What one has done is to replace :

$$\hat{k}_{j_m j_l} \equiv \frac{k_{j_m j_l}}{w_{j_m}} \rightarrow \int dt_m \frac{\delta''(t_m - t_l)}{w(t_m)} = \left[ \frac{1}{w(t_l)} \right]_{,t_l t_l} \dots \quad (21)$$

in all the equations in the continuum limit. One has smeared out the delta functions in the denominators of (14) using the function  $w(t)$ . If one had not smeared out the delta functions one would have encountered ill-defined expressions. From now on we set  $\hbar = 1$ .

These are the quantum corrections to the classical solution  $\rho = \partial^2 x / \partial t^2$  where  $x(r, t)$  is given in (14). These are the continuum limits of eqs-(82-86) of [3]. It is important to realize that one must not add quantum corrections to the  $\varphi, d(t)$  appearing in the terms  $\exp[r \sum \varphi]$  and  $x_o$  of (14). The former are two arbitrary functions which parametrize the space of solutions. It is  $\rho$  and  $x$  which acquire quantum corrections given by (19,20) through the  $w(t)$  terms and, as such,  $\rho, x$  should be seen as quantum operators acting on the Hilbert space of states. Upon quantization,  $\hbar$  appears and associated with Planck's constant a new parametric function has to appear :  $w(t)$ . One has to incorporate also the coupling constant  $\beta$  in all of the equations . This is achieved by rescaling the continuous Cartan matrix by a factor of  $\beta$  so that  $\partial^2 x / \partial t^2$  and  $\partial^2 x_o / \partial t^2$  are rescaled by a factor of  $\beta$ ; i.e.  $r\varphi$  acquires a factor of  $\beta$  and  $d(t) \rightarrow d(t)^\beta$ . Since  $\beta$  is pure imaginary, for convergence purposes in the  $r = \infty$  region we must have that  $\beta\varphi < 0 \Rightarrow \varphi = i\varphi$  also. In the rest of this section we will work without the  $\beta$  factors and only reinsert them at the end of the calculations. There is nothing unphysical about this value of  $\beta$  as we said earlier.

One of the integrals of motion is the energy. The continuous Toda chain is an exact integrable system in the sense that it possesses an infinite number of functionally independent integrals of motion :  $I_n(p, \rho)$  in involution. i.e. The Poisson brackets amongst  $I_n, I_m$  is zero. Since these are integrals of motion, they do not depend on  $r$ . These integrals can be evaluated most easily in the asymptotic region  $r \rightarrow \infty$ . This was performed in [2] for the case that  $\varphi(t)$  was a negative real valued function which simplified the calculations. For this reason the energy eigenvalue given in [2] must now be rescaled by a factor of  $\beta^2$  :

$$E = \beta^2 \int_0^{2\pi} dt \left( \int^t dt' \varphi(t') \right)^2. \quad (22)$$

where we have chosen the range of the  $t$  integration to be  $[0, 2\pi]$ . Since  $\beta\varphi < 0 \Rightarrow \beta^2\varphi^2 > 0$  and the energy is positive. We insist, once more, that  $t$  is a parameter which is not the physical time and that  $\varphi(t)$  does not acquire quantum corrections. The latter integral (22) is the eigenvalue of the Hamiltonian which is one of the Casimir operators for the irreducible representations of  $A_N$  in the  $N \rightarrow \infty$  limit.

We can borrow now the results by [10,11] on the quasi-finite highest weight irreducible representations of  $W_{1+\infty}$  and  $W_\infty$  algebras. The latter is a subalgebra of the former. For each highest weight state,  $|\lambda\rangle$  parametrized by a complex number  $\lambda$  the authors [10,11] constructed representations consisting of a finite number of states at each energy level by successive application of ladder-like operators. A suitable differential constraint on the generating function  $\Delta(x)$  for the highest weights  $\Delta_k^\lambda$  of the representations was necessary in order to ensure that, indeed, one has a finite number of states at each level. The highest weight states are defined :

$$W(z^n D^k) |\lambda\rangle = 0. \quad n \geq 1, k \geq 0. \quad W(D^k) |\lambda\rangle = \Delta_k^\lambda |\lambda\rangle. \quad k \geq 0. \quad (23)$$

The  $W_{1+\infty}$  algebras can be defined as central extensions of the Lie algebra of differential operators on the circle.  $D \equiv z d/dz$ .  $n \in \mathbb{Z}$  and  $k$  is a positive integer. The generators of

the  $W_{1+\infty}$  algebra are denoted by  $W(z^n D^k)$  and the  $W_\infty$  generators are obtained from the former :  $\tilde{W}(z^n D^k) = W(z^n D^{k+1})$ ; where  $(n, k \in \mathbb{Z}, k \geq 0)$ . (There is no spin one current). The generating function  $\Delta(x)$  for the weights is :

$$\Delta(x) = \sum_{k=0}^{k=\infty} \Delta_k^\lambda \frac{x^k}{k!}. \quad (24)$$

where we denoted explicitly the  $\lambda$  dependence as a reminder that we are referring to the highest weight state  $|\lambda\rangle$  and satisfies the differential equation required for quasi-finiteness :

$$b(d/dx)[(e^x - 1)\Delta(x) + C] = 0. \quad b(w) = \prod (w - \lambda_i)^{m_i}, \quad \lambda_i \neq \lambda_j. \quad (25)$$

$b(w)$  is the characteristic polynomial.  $C$  is the central charge and the solution is :

$$\Delta(x) = \frac{\sum_{i=1}^K p_i(x) e^{\lambda_i x} - C}{e^x - 1}. \quad (26)$$

The generating function for the  $W_\infty$  case is  $\tilde{\Delta}(x) = (d/dx)\Delta(x)$  and the central charge is  $c = -2C$ .

The Verma module is spanned by the states :

$$|v_\lambda\rangle = W(z^{-n_1} D^{k_1}) W(z^{-n_2} D^{k_2}) \dots W(z^{-n_m} D^{k_m}) |\lambda\rangle. \quad (27)$$

The energy level is  $\sum_{i=1}^{i=m} n_i$ . For further details we refer to [10,11]. Highest weight unitary representations for the  $W_\infty$  algebra obtained from field realizations with central charge  $c = 2$  were constructed in [10].

The weights associated with the highest weight state  $|\lambda\rangle$  will be obtained from the expansion in (24). In particular, the "energy" operator acting on  $|\lambda\rangle$  will be :

$$W(D)|\lambda\rangle = \Delta_1^\lambda |\lambda\rangle. \quad (28)$$

$L_o = -W(D)$  counts the energy level :  $[L_o, W(z^n D^k)] = -n W(z^n D^k)$ .

As an example we can use for  $\Delta(x)$  the one obtained in the free-field realization by free fermions or **bc** ghosts [10]

$$\Delta(x) = C \frac{e^{\lambda x} - 1}{e^x - 1} \Rightarrow \partial \Delta / \partial \lambda = C \frac{x e^{\lambda x}}{e^x - 1}. \quad (29)$$

The second term is the generating function for the Bernoulli polynomials :

$$\frac{x e^{\lambda x}}{e^x - 1} = 1 + (\lambda - 1/2)x + (\lambda^2 - \lambda + 1/6) \frac{x^2}{2!} + (\lambda^3 - 3/2\lambda^2 + 1/2\lambda) \frac{x^3}{3!} + \dots \quad (30)$$

Integrating (30) with respect to  $\lambda$  yields back :

$$\Delta(x) = C \frac{e^{\lambda x} - 1}{e^x - 1} = \sum_{k=0} \Delta_k \frac{x^k}{k!}. \quad (31)$$

The first few weights (modulo a factor of  $C$ ) are then :

$$\Delta_0 = \lambda. \Delta_1 = (1/2)(\lambda^2 - \lambda). \Delta_2 = (1/3)\lambda^3 - (1/2)\lambda^2 + (1/6)\lambda \dots \quad (32)$$

The generating function for the  $W_\infty$  case is  $\tilde{\Delta}(x) = \frac{d\Delta(x)}{dx} \Rightarrow \tilde{\Delta}_k^\lambda = \Delta_{k+1}^\lambda$ .

$\tilde{\Delta}_1^\lambda = \Delta_2^\lambda$  is the weight associated with the "energy" operator. In the ordinary string,  $W_2$  algebra, the Hamiltonian is related to the Virasoro generator,  $H = L_0$  and states are built in by applying the ladder-like operators to the highest weight state, the "vacuum". In the  $W_{1+\infty}$ ,  $W_\infty$  case it is not longer true that the Hamiltonian (52) can be written exactly in terms of the zero modes of the  $W_2$  generator, once the realization of the  $W_\infty$  algebra is given in terms of the continous Toda field given by [2] :

$$\tilde{W}_2^+ = \int^{t_0} dt_1 \int^{t_1} dt_2 \exp[-\Theta(z, \bar{z}; t_1)] \frac{\partial}{\partial z} \exp[\Theta(z, \bar{z}; t_1) - \Theta(z, \bar{z}; t_2)] \frac{\partial}{\partial z} \exp[\Theta(z, \bar{z}; t_2)]. \quad (33)$$

The chiral generator has the form  $W_h^+[\partial\rho/\partial z \dots \partial^h \rho/\partial z^h]$  [2] and the similar expression for the antichiral generator  $W_h^-$  is obtained by replacing  $\partial_z \rightarrow \partial_{\bar{z}}$  in (33). After a dimensional reduction from  $D = 3 \rightarrow D = 2$  is taken,  $r = z + \bar{z}$ , one has :

$$\tilde{W}_2(r, t_0) = \int^{t_0} dt_1 \int^{t_1} dt_2 \exp[-\rho(r, t_1)] \frac{\partial}{\partial r} \exp[\rho(r, t_1) - \rho(r, t_2)] \frac{\partial}{\partial r} \exp[\rho(r, t_2)]. \quad (34)$$

When  $\rho(r, t)$  is quantized in eq-(19,20) it becomes an operator,  $\hat{\rho}(r, t)$ , acting on a suitable Hilbert space of states, say  $|\rho\rangle$ , and in order to evaluate (34) one needs to perform the highly complicated Operator Product Expansion between the operators  $\hat{\rho}(r, t_1), \hat{\rho}(r, t_2)$ . Since these are no longer free fields it is no longer trivial to compute per example the operator products :

$$\frac{\partial \rho}{\partial r} \cdot e^\rho \cdot e^{\rho(r, t_1)} \cdot e^{\rho(r, t_2)} \dots \quad (35)$$

Quantization deforms the classical  $w_\infty$  algebra into  $W_\infty$  [17,18]. For a proof that the  $W_\infty$  algebra is the Moyal bracket deformation of the  $w_\infty$  see [18]. Later in [19] we were able to construct the non-linear  $\hat{W}_\infty$  algebras from non-linear integrable deformations of Self Dual Gravity in  $D = 4$ . Since the  $w_\infty$  algebra has been effectively quantized the expectation value of the  $\tilde{W}_2$  operator at tree level,  $\langle \rho | \tilde{W}_2 | \rho \rangle$ , is related to the  $\tilde{W}_2(\text{classical})$  given by (34). One can evaluate all expressions in the  $r = \infty$  limit ( and set  $d(t) = 1$  for convenience. The expectation value  $\langle \rho | \hat{W}_2(\hat{\rho}) | \rho \rangle (\varphi(t))$  gives in the  $r = \infty$  limit, after the dimensional reduction and after using the asymptotic limits :

$$\frac{\partial \rho}{\partial r} = \varphi. \quad \frac{\partial^2 \rho}{\partial r^2} = \frac{\partial^2 e^\rho}{\partial t^2} \rightarrow 0. \quad r \rightarrow \infty \quad (36)$$



$$\lim_{r \rightarrow \infty} \langle \rho | \hat{W}_2 | \rho \rangle = \int^{t_o} dt_1 \varphi(t_1) \int^{t_o} dt_1 \varphi(t_1). \quad (37)$$

after the normalization condition is chosen :

$$\langle \rho' | \rho \rangle = \delta(\rho' - \rho). \quad \langle \rho | \rho \rangle = 1 \quad (38)$$

. We notice that eq-(37) is the same as the integrand (22); so integrating (37) with respect to  $t_o$  yields the energy. It is useful to recall the results from ordinary  $2D$  conformal field theory : given the holomorphic current generator of two-dimensional conformal transformations,  $T(z) = W_2(z)$ , the mode expansion is :

$$W_2(z) = \sum_m W_2^m z^{-m-2} \Rightarrow W_2^m = \oint \frac{dz}{2\pi i} z^{m+2-1} W_2(z). \quad (39)$$

the closed integration contour encloses the origin. When the closed contour surrounds  $z = \infty$ . This requires performing the conformal map  $z \rightarrow (1/z)$  and replacing :

$$z \rightarrow (1/z). \quad dz \rightarrow (-dz/z^2). \quad W_2(z) \rightarrow (-1/z^2)^2 W_2(1/z). \quad (40)$$

in the integrand.

There is also a 1 - 1 correspondence between local fields and states in the Hilbert space :

$$|\phi \rangle \leftrightarrow \lim_{z, \bar{z} \rightarrow 0} \hat{\phi}(z, \bar{z}) |0 \rangle. \quad (41)$$

This is usually referred as the  $|in \rangle$  state. A conformal transformation  $z \rightarrow 1/z; \bar{z} \rightarrow 1/\bar{z}$ : defines the  $\langle out|$  state at  $z = \infty$

$$\langle out| = \lim_{z, \bar{z} \rightarrow 0} [(-1/z^2)^h (-1/\bar{z}^2)^{\bar{h}} \hat{\phi}(1/z, 1/\bar{z}) |0 \rangle]^+. \quad (42)$$

where  $h, \bar{h}$  are the conformal weights of the field  $\phi(z, \bar{z})$ .

The analog of eqs-(42) is to consider the state parametrized by  $\varphi(t)$  :

$$|\rho \rangle_{\varphi(t)} = \lim_{r \rightarrow \infty} |\rho(r, t) \rangle \equiv |\rho(out) \rangle. \quad |\rho \rangle_{-\varphi(t)} = \lim_{r \rightarrow -\infty} |\rho(r, t) \rangle \equiv |\rho(in) \rangle. \quad (43)$$

since the continuous Toda equation is symmetric under  $r \rightarrow -r \Rightarrow \rho(-r, t)$  is also a solution and it's obtained from (14) by setting  $\varphi \rightarrow -\varphi$  to ensure convergence at  $r \rightarrow -\infty$ .

The state  $|\rho \rangle$  is parametrized in terms of  $\varphi$  and for this reason one should always write it as  $|\rho \rangle_{\varphi}$ . What is required now is to establish the correspondence (a functor) between the representation space realized in terms of the continuous Toda field and that representation ( the Verma module) built from the highest weight  $|\lambda \rangle$

$$\langle \lambda | \tilde{W}(D) | \lambda \rangle = \tilde{\Delta}_1^\lambda \equiv \Delta_2^\lambda \leftrightarrow \langle \rho | \hat{W}_2[\hat{\rho}(r, t)] | \rho \rangle. \quad (44)$$

What is required then is to integrate with respect to  $t_o$ , to extract the zero mode piece of the  $\tilde{W}_2$  operator via a contour integral around the origin , :

$$\tilde{\Delta}_1^\lambda = \langle W_2^0 \rangle \leftrightarrow \int_0^{2\pi} dt_o \langle \rho | \oint \frac{dz}{2\pi i} \oint \frac{d\bar{z}}{2\pi i} z\bar{z} \hat{W}_2[\rho(z, \bar{z}, t)] | \rho \rangle . \quad (45)$$

the contour integral also could be performed around infinity :  $z = 0 \rightarrow 1/z = \infty$  if one wishes. The expectation value of :  $\langle \rho | \hat{W}_2 | \rho \rangle$  in the dimensionally-reduced case, depends on  $r = z + \bar{z} \Rightarrow \langle \rho | \hat{W}_2 | \rho \rangle (r, t)$ . The latter is a function which can be expanded in powers of  $z + \bar{z} : \sum a_n (z + \bar{z})^{-n}$  and the integral (45) can be computed.

Rigorously speaking one must also include the realization of the anti-chiral algebra  $\bar{W}_{1+\infty}$  [10,11] in terms of  $\bar{W}(\bar{z}^n \bar{D}^k)$  which yields the weights  $\bar{\Delta}_k^\lambda$ . Thus the r.h.s of (45) involves both types of weights.

Also required is to introduce a family of functions  $\varphi_\lambda(t)$  parametrized by  $\lambda$ ; i.e to each  $|\lambda\rangle \leftrightarrow |\varphi_\lambda\rangle$  so that  $\langle \rho | \hat{W} | \rho \rangle (r, t)$  is functionally dependent on  $\varphi_\lambda(t)$ . Eq-(45) is an integral equation relating  $\tilde{\Delta}_1^\lambda$  to the family of functions  $\varphi_\lambda(t)$  linking in this way the realizations in terms of the continuous Toda field and the highest weight  $W_\infty$  irrepresentations.

The l.h.s of (45) corresponds to the action of  $\tilde{W}(D)$  on  $|\lambda\rangle$  in the language of [10,11]. The r.h.s of (45) corresponds indeed to extract the zero mode part of the  $\tilde{W}_2$  generator, in the realization of the  $W_\infty$  algebra in terms of the continuous Toda field, after the dimensional reduction  $z + \bar{z} = r$  has taken place. The other weights,  $\tilde{\Delta}_k^\lambda$  and the anti-chiral ones, are also obtained from the zero modes of Saveliev's realization of the chiral and antichiral  $W_\infty$  algebras,

$$W_h^+ [\partial \rho / \partial z \dots \partial^h \rho / \partial z^h] . W_h^- [\partial_z \rho \rightarrow \partial_{\bar{z}} \rho] . \partial W^+ / \partial \bar{z} = 0 . \partial W^- / \partial z = 0 \quad (46)$$

.

$$\int_0^{2\pi} dt_o \langle \rho | \oint \frac{dz}{2\pi i} \oint \frac{d\bar{z}}{2\pi i} z^{h-1} \bar{z}^{\bar{h}-1} W_h | \rho \rangle \leftrightarrow \langle \lambda | \tilde{W}(D^k) | \lambda \rangle = \tilde{\Delta}_k^\lambda . \quad (47)$$

where  $k \geq 1$ .  $\langle \rho | W_h | \rho \rangle (r, t)$  will depend on  $(\varphi_\lambda(t))$  after the dimensional reduction takes place.

Thus Eqs-(45,47) are the equations we were looking for.

Therefore, to conclude, eqs- (45,47) are the eigenvalue equations which determines the very intricate relationship between  $\varphi_\lambda(t)$  and the weights  $\tilde{\Delta}_k^\lambda$ . Given a quasi-finite highest-weight irreducible representation; i.e. given the set  $\tilde{\Delta}^\lambda(x) \Rightarrow \Delta_k^\lambda; C, b(w), \chi, \dots$  one can from eq-(45,47) determine  $\varphi_\lambda(t)$  as a family of functions parametrized by  $\lambda$ . It is essential to maintain that  $\varphi(t) < 0$  in order to use the asymptotic  $r = \infty$  expression for the energy (22). Since  $\lambda$  is a continuous parameter the energy spectrum (22) is continuous in general. Below we will study a simple case when one has a discrete spectrum characterized by the positive integers  $n \geq 0$ . It is important to realize that (37) is equal to the expression for the classical energy density (22) not only at  $r = \infty$  but for other values of  $r$  so that :

$$\hat{H}[W_2[\hat{\rho}(\varphi_\lambda(t), d_\lambda(t))]] | \rho \rangle = E(\varphi_\lambda) | \rho \rangle . \quad (48)$$

we recall that  $E(\lambda)$  is not always given by eq-(22). The latter equation is valid only for a very special case when the function  $\varphi(t)$  is real and negative.

The functions  $\varphi(t) < 0$  in [2] were taken to belong to the space of trigonometric polynomials in the circle. One may expand :

$$\varphi_\lambda(t) = \sum_{m=0} A_m(\lambda) \cos(mt) + B_m(\lambda) \sin(mt). \quad (49)$$

and may take  $A_m, B_m$  to be functions of the  $\lambda$  parameter characterizing the representations (like the weights). If this is the case, having an infinite family of functions in  $\lambda$ ,  $\tilde{\Delta}_k^\lambda$ .  $k = 1, 2, \dots$ , the integral equations (45,47) for  $h = 2, 3, 4, \dots$ , will be sufficient to specify  $A_m(\lambda), B_m(\lambda)$ .  $m = 0, 1, 2, \dots$ . In this fashion one can determine  $\varphi_\lambda(t)$  and establish the  $|\lambda\rangle \rightarrow |\varphi_\lambda\rangle$  correspondence. Therefore, plugging (49) into (45,47) determines the coefficients  $A_m, B_m$  as functions of  $\lambda$  when  $h = 2, 3, \dots$ . Eqs-(45,47) are difficult to solve in general. There is a special case when we can solve it.

Below we will study a simple case when one has a discrete spectrum characterized by the positive integers  $n \geq 0$ . We shall restore now the coupling  $\beta^2 < 0$  given in (13). A simple fact which allows for the possibility of discrete energy states is to use the analogy of the Bohr-Sommerfeld quantization condition for periodic system. It occurs if one opts to choose for the quantity  $\exp[\beta\varphi(t)r] \equiv \exp[i\Omega r]$  which appears in (14);  $\Omega$  is the frequency parameter ( a constant ). If the dynamical system is periodic in the variable  $r$  with periods  $2\pi/\Omega$ , a way to quantize the values of  $\Omega$  in units of  $n$  is to recur to the Bohr-Sommerfeld quantization condition for a periodic orbit :

$$J = \oint pdq = nh \quad (50a)$$

which reflects the fact that upon emission of a quanta of energy  $(h/2\pi)\Omega$  the change in the energy level as a function of  $n$  is [16] (set  $h/2\pi = 1$ ) :

$$\partial E / \partial n = \Omega = \frac{2}{3}(2\pi)^3 \Omega \partial \Omega / \partial n. \Rightarrow$$

$$\Omega(n) = \frac{3}{2(2\pi)^3} n. \quad (50b)$$

Hence the energy is

$$E = \frac{3}{4}(2\pi)^{-3} n^2. \quad (51)$$

which is reminiscent of the rotational energy levels  $E \sim h^2 l(l+1)$  of a rotor in terms of the angular momentum quantum numbers  $l = 0, 1, \dots$ . In order to have a proper match of dimensions it is required to insert the membrane tension as it happens with the string.

Saveliev [2] chose the  $\varphi(t)$  in (14) to be negative real functions to assure that the potential term in the Hamiltonian vanished at  $r \rightarrow \infty$  and arrived at (22). In case that the functions  $\varphi(t)$  are no longer  $< 0$ ; i.e when  $\beta\varphi r$  is no longer a real valued quantity  $< 0$ , the asymptotic formula (22) will no longer hold and one will be forced to perform the very complicated integral !

$$\mathcal{H} = \int dt \left[ -\frac{1}{2} \beta^2 (\partial p / \partial t)^2 + \left( \frac{m^2}{\beta^2} \right) \exp [\beta \partial^2 x / \partial t^2] \right]. \quad (52)$$

where  $p = \beta \partial x / \partial r$  is the generalized momentum corresponding to  $\rho \equiv \beta \partial^2 x / \partial t^2$ , and  $\mu^2 \equiv (\frac{m^2}{\beta^2})$  is the perturbation theory expansion parameter discussed in [3]. Without loss of generality it can be set to one. Nevertheless, eqs-(50,51) are still valid. One just needs to evaluate the Hamiltonian at  $\Omega r = 2\pi p$  where  $p$  is a very large integer  $p \rightarrow \infty$  and take  $d(t) = 0$  in (15,16) :

$$\exp[\partial^2 x / \partial t^2] \rightarrow d(t) \exp[i 2\pi p] = 0. (\partial p / \partial t)^2 \rightarrow \left( \int \varphi dt' \right)^2 \dots$$

recovering (22) once again.

Are there zero energy solutions ?. If one naively set  $\varphi(t) \equiv 0$  in (14) or set  $n = 0$  in (51) one would get a zero energy. However eqs-(14,15) for the most part will be singular and this would be unacceptable. One way zero energy states could be obtained is by choosing  $\varphi(t), d(t)$  appropriately so that (52) is zero. Since one has one equation and two functions to vary presumably there should be an infinite number of solutions of zero energy.

Solving (48) is analogous to solving an Schroedinger-like equation. Concentrating on the case that  $\varphi(t) < 0$ ; the wave functional is defined :  $\Psi[\rho, t] \equiv \langle \rho | \Psi \rangle$  where the state  $|\rho \rangle_\varphi$  has an explicit dependence on  $\varphi$  which also depends on  $\lambda$  as shown in (45,47). Upon replacing (in units  $\hbar/2\pi = 1$ )

$\partial p / \partial t \rightarrow i(\partial / \partial t \delta / \delta \rho)$  as an operator acting on the  $\Psi$ , the time-independent Schroedinger-like equation for the wave functional becomes :

$$[(\partial^2 / \partial t^2 \delta^2 / \delta \rho^2) + \exp \rho] \Psi[\rho, t] = \frac{E(\varphi)}{2\pi} \Psi[\rho, t]. \quad (53)$$

One could have for an eigenvalue-density,  $\gamma(t)$ , in the r.h.s of (53) other values than the particular constant  $E/2\pi$  if one wishes. In this case the solutions are more complex. The action functional is :

$$\int dt \int \mathcal{D}\rho dr \Psi^+ (i \frac{\partial}{\partial r} - \mathcal{H}) \Psi. \quad (54)$$

$\mathcal{D}\rho$  is the functional integration measure;  $r$  is the variable linked with the physical time and the on-shell condition is just (53). This is the second-quantization of the physical quantities.  $\rho(r, t)$  has already been first-quantized in (19,20). One should not interpret  $\Psi$  as a probability amplitude but as a field operator ( "membrane" field) which creates a continuous Toda field in a given quantum state  $|\rho \rangle_\varphi$  associated with the classical configuration configuration given by eq-(14) in terms of  $\varphi_\lambda(t)$ .

One can expand  $|\Psi \rangle$  in an infinite dimensional basis spanned by the Verma module (27) associated with the state  $|\lambda \rangle$ . Given a vector  $v_\lambda \in \mathcal{V}$  ( Verma module) one has :

$$|\Psi \rangle = \sum_{v_\lambda} \langle v_\lambda | |\Psi \rangle | v_\lambda \rangle . v_\lambda \in \mathcal{V} \quad (55)$$

This is very reminiscent of the string-field  $\Phi[X(\sigma)] = \langle X | \Phi(x_o) \rangle$  where  $x_o$  is the center of mass coordinate of the string and the state  $|\Phi(x_o)\rangle$  is comprised of an infinite array of point fields :

$$|\Phi(x_o)\rangle = \phi(x_o)|0\rangle + A_\mu(x_o)a_1^{\mu+}|0\rangle + g_{\mu\nu}a_1^{\mu+}a_1^{\nu+}|0\rangle + \dots \quad (56)$$

where the first field is the tachyon, the second is the massless Maxwell, the third is the massive graviton.... The oscillators play the role of ladder-like operators acting on the "vacuum"  $|0\rangle$  in the same manner that the Verma module is generated from the highest weight state  $|\lambda\rangle$  by successive application of a string of  $W(z^{-n}D^k)$  operators acting on  $|\lambda\rangle$ . The state  $|\rho(r, t)\rangle_\varphi$  is the relative of the string state  $|X(\sigma_1, \sigma_2)\rangle$  whereas  $|\Psi\rangle$  is the relative of the string field state  $|\Phi\rangle$ . The Schroedinger-like equation is of the form :

$$[\partial_t^2 \partial_y^2 + e^y]\Psi(y, t) = E\Psi(y, t). \quad (57)$$

A change of variables  $x = 2e^{y/2}$  converts (57) into Bessel's equation after one sets  $\Psi(y, t) = e^t \Phi(y)$  or equal to  $e^{-t} \Phi(y)$  :

$$(x^2 \partial_x^2 + x \partial_x + x^2 - 4E)\Phi(x) = 0. \quad (58)$$

and whose solution is  $\Phi(x) = c_1 \mathcal{J}_\nu(x) + c_2 \mathcal{J}_{-\nu}(x)$  where  $\nu \equiv 2\sqrt{E}$  and  $c_1, c_2$  constants. The wavefunctional is then :

$$\Psi[\rho, t] = \int dr e^t [c_1 \mathcal{J}_\nu(2e^{\rho(r,t)/2}) + c_2 \mathcal{J}_{-\nu}(2e^{\rho(r,t)/2})] \quad (59)$$

or the other solution involving  $e^{-t}$ ....

One may notice that discrete energy level (in suitable units such as  $\nu = 2\sqrt{E} = n$ ) solutions are possible. Earlier we saw in (51) that  $E(n) \sim n^2$  so  $\sqrt{E} \sim n$ . Therefore setting  $2\alpha e^{y/2\alpha} = x$  where  $\alpha$  is a suitable constant allows to readjust  $\nu = 2\sqrt{\alpha}E = n$ . The Bessel functions will have nodes at very specific points. The solutions in this case will be given in terms of  $\mathcal{J}_n$  and the modified Bessel function of the second kind,  $K_n$ . These solutions are tightly connected with the boundary conditions of the wave-functional.

It has been argued that a four dimensional anti-deSitter spacetime,  $AdS_4$ , whose boundary is  $S^2 \times S^1$ , could be realized as a membrane at the end of the universe. In particular, singleton field theory can be described on the boundary of  $AdS_4$  where singletons are the most fundamental representations of the de Sitter groups [20]. Moreover, on purely kinematical grounds, infinitely many massless states of all spins (massless in the anti-de-Sitter sense) can be constructed out of just two singletons (preons). In particular, the  $d = 4$   $N = 8$  Supersingleton field theory formulated on the boundary of  $AdS_4$  bears a connection with the supermembrane moving on  $AdS_4 \times S^7$ . The rigid  $OSp(8|4)$  symmetry acts as the superconformal group on the boundary  $S^2 \times S^1$  [20]. In view of this it is important to study if there is any connection between the wave-functional behaviour at the boundaries and singleton field theory. The supersymmetric Toda equation has also been discussed by [2,3]. Roughly speaking, a membrane is comprised of an infinite number of strings. Thus the membrane can be seen as a coherent state of an infinite number of

strings . This is reminiscent of the Sine-Gordon soliton being the fundamental fermion of the massive Thirring model, a quantum lump [16]. The lowest fermion-antifermion bound state (soliton-antisoliton doublet) is the fundamental meson of Sine-Gordon theory. Higher level states are built from excitations of the former in the same way that infinitely many massless states can be built from just two singletons.

Perhaps the most relevant physical applications of the membrane quantization program will be in the behaviour of black hole horizons [8]. These have been described in terms of a dynamical surface whose quantum dynamics is precisely that of a relativistic membrane. Thermodynamical properties like the entropy and temperature of the black hole were derived in agreement with the standard results. Results for the level structure of black holes were given. A "principal" series of levels was found corresponding to the quantization of the area of the horizon in units of the "area quantum" :  $A = nA_o$ .  $A_o = 8\pi$ . From each level of this principal series starts a quasi-continuum of levels due to the membrane's excitations.

Secondly, the ordinary bosonic string has been found to be a special vacua of the  $N = 1$  superstring [14]. It appears that there is a whole hierarchy of string theories :  $w_2$  string is a particular vacua of the  $w_3$  string and so forth.....If this is indeed correct one has then that the (super) membrane, viewed as noncritical  $W_\infty$  string theory, is, in this sense, the universal space of string theory. The fact, advocated by many, that a Higgs symmetry-breakdown-mechanism of the infinite number of massless states of the membrane generates the infinite massive string spectrum fits within this description.

Finally, we hope that the essential role that Self Dual  $SU(\infty)$  Yang-Mills theory has played in the origins of the membrane-Toda theory, will shed more light into the origin of duality in string theory [7].

ACKNOWLEDGEMENTS. We thank M.V. Saveliev for many helpful suggestions concerning the quantization program. To Bob Murray for refreshing to us the appropriate change of variables in connection to Bessel's differential equation and to the Towne family for their kind and warm hospitality in Austin, Texas.

## REFERENCES

1. C. Castro : "  $D = 11$  Supermembrane Instantons,  $W_\infty$  Strings and the Super Toda Molecule " IAEC-12-94.  
submitted to the Jour. of Chaos, Solitons and Fractals. hep-th-lanl-9412160.
- 2-M.V. Saveliev : Theor. Math. Physics **vol. 92.** no.3 (1992) 457.
- 3.A.N.Leznov, M.V.Saveliev, I.A.Fedoseev : Sov. J. Part. Nucl. **16** no.1 (1985) 81.  
A.N.Leznov, M.V.Saveliev : "Group Theoretical Methods for Integration of Nonlinear Dynamical Systems " Nauka, Moscow, 1985.
- 4.R.Blumenhagen, W. Eholzer, A. Honecker, K. Hornfeck, R. Hubel : " Unifying  $W$  algebras".  
Bonn-TH-94-01 April-94. hep-th-lanl-9404113. Phys. Letts. **B 332** ( 1994) 51.
- 5.H.Lu, C.N. Pope, X.J. Wang: Int. J. Mod. Phys. Lett. **A9** (1994) 1527.  
H.Lu, C.N. Pope, X.J. Wang, S.C. Zhao : "Critical and Non-Critical  $W_{2,4}$  Strings".  
CTP-TAMU-70-93. hep-th-lanl-9311084.
- H.Lu, C.N. Pope, K. Thielemans, X.J. Wang, S.C. Zhao : "Quantising Higher-Spin String Theories "

- CTP-TAMU-24-94. hep-th/9410005.
6. E. Bergshoeff, H. Boonstra, S. Panda, M. de Roo : Nucl. Phys. **B 411** (1994) 717.
  7. M.Duff, R. Minasian : "Putting String/String Duality to the Test"  
CTP-TAMU-16/94. hep-th/9406198.
  8. M. Maggiore : "Black Holes as Quantum Membranes : A Path Integral Approach "  
hep-th/9404172.
  9. M.A.C. Kneipp, D.I. Olive : Nucl.Phys. **B 408** (1993) 565.
  10. H.Awata, M.Fukuma, Y.Matsuo, S.Odake : "Representation Theory of the  $W_{1+\infty}$  Algebra".  
RIMS-990 Kyoto preprint , Aug.1994.  
S.Odake : Int.J.Mod.Phys. **A7** no.25 (12) 6339.
  11. V.Kac, A. Radul : Comm. Math. Phys. **157** (1993) 429.
  12. J. de Boer : "Extended Conformal Symmetry in Non-Critical String Theory" .  
Doctoral Thesis. University of Utrecht, Holland. (1993).
  13. J.de Boer, J. Goeree : Nucl. Phys. **B 381** (1992) 329.
  14. N. Berkovits, C.Vafa : Mod. Phys. Letters **A9** (1994) 653.
  15. P. Bouwknegt, K. Schoutens : Phys. Reports **223** (1993) 183.
  16. S. Coleman : "Aspects of Symmetry " Selected Erice Lectures.  
Cambridge Univ. Press. (1989). Chapter 6, page 239.
  17. C.Pope, L.Romans, X. Shen : Phys. Lett. **B 236** (1990) 173.
  18. I.Bakas, B.Khesin, E.Kiritsis : Comm. Math. Phys. **151** (1993) 233.
  19. D. Fairlie, J. Nuyts : Comm. Math. Phys. **134** (1990) 413.
  20. C.Castro : "Non-linear  $W_\infty$  Algebras from Non-linear Integrable Deformations of Self Dual Gravity" hep-th/9409197.
  20. M.Duff : Class. Quant. Grav. **5** (1988) 189.
  - E.Bergshoeff, A.Salam, E.Sezgin ,Y.Tanii : Nucl. Phys. **B 305** (1988) 497.
  - E.Bergshoeff, E.Sezgin ,P.Townsend : Phys. Letts. **B 189** (1987) 75.